

REGULARITY PROBLEM FOR THE NEMATIC LCD SYSTEM WITH Q-TENSOR IN \mathbb{R}^3

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ABSTRACT. We study the regularity problem of a nematic liquid crystal model with local configuration represented by Q-tensor in three dimensions. It was an open question whether the classical Prodi-Serrin condition implies regularity for this model. Applying a wavenumber splitting method, we show that a solution does not blow-up under certain extended Beale-Kato-Majda condition solely imposed on velocity. This regularity criterion automatically implies that the classical Prodi-Serrin or Beale-Kato-Majda condition prevents blow-up of solutions.

KEY WORDS: Nematic liquid crystals; Q-tensor configuration; regularity; wavenumber splitting.

CLASSIFICATION CODE: 35B44, 35Q35, 76A15, 76D03.

1. INTRODUCTION

Considered here is a hydrodynamic model of nematic liquid crystals proposed by Beris and Edwards [6], where the local configuration of the crystal is represented by the Q -tensor $\mathbb{Q} = \mathbb{Q}(x, t)$. The evolution of the crystal flow is governed by

$$(1.1) \quad \begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= \nu \Delta u + \nabla \cdot \Sigma(\mathbb{Q}), \\ \mathbb{Q}_t + (u \cdot \nabla)\mathbb{Q} - \mathbb{S}(\nabla u, \mathbb{Q}) &= \mu \Delta \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})] \\ \nabla \cdot u &= 0, \end{aligned}$$

with $(x, t) \in \mathbb{R}^n \times (0, \infty)$. In the equations, the Q -tensor \mathbb{Q} describes the ordering of the molecule, u is the fluid velocity, p is the fluid pressure and F denotes a potential function which will be described later. The parameter ν denotes the kinematic viscosity coefficient of the fluid, and μ stands for the elasticity of the molecular orientation field. The tensor $\mathbb{Q} \in \mathbb{R}_{\text{sym},0}^{3 \times 3}$ is symmetric and traceless. Other notations are introduced in the following. The operator \mathcal{L} is defined as

$$\mathcal{L}[\mathbb{A}] = \mathbb{A} - \frac{1}{3} \text{tr}[\mathbb{A}] \mathbb{I}$$

which represents the projection onto the space of traceless matrices. The tensors Σ and \mathbb{S} are given by

$$(1.2) \quad \Sigma(\mathbb{Q}) = 2\xi \mathbb{H} : \mathbb{Q} \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) - \xi \left[\mathbb{H} \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) - \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) \mathbb{H} \right] - (\mathbb{Q} \mathbb{H} - \mathbb{H} \mathbb{Q}) - \nabla \mathbb{Q} \otimes \nabla \mathbb{Q},$$

$$(1.3) \quad \mathbb{S}(\nabla u, \mathbb{Q}) = (\xi D(u) + \Omega(u)) \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) + \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) (\xi D(u) - \Omega(u)) - 2\xi \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) \mathbb{Q} : \nabla u,$$

with $(\nabla \mathbb{Q} \otimes \nabla \mathbb{Q})_{ij} = \partial_i \mathbb{Q}_{\alpha\beta} \partial_j \mathbb{Q}_{\alpha\beta}$. Note that we use the summation convention for repeated indices here and through the rest of the paper. In the above equations, D and Ω denote the symmetric and skew-symmetric parts of the rate of strain tensor, respectively

$$D(u) = \frac{1}{2}(\nabla u + \nabla^t u), \quad \Omega(u) = \frac{1}{2}(\nabla u - \nabla^t u).$$

While \mathbb{H} is obtained through the variational derivative of the free energy under the constraint that \mathbb{Q} is symmetric and traceless, as

$$\mathbb{H} = \Delta \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})],$$

where $F(\mathbb{Q})$ denotes the bulk potential function. In this work, we take the Landau-de Gennes form

$$(1.4) \quad F(\mathbb{Q}) = \frac{a}{2}|\mathbb{Q}|^2 + \frac{b}{3}\text{tr}[\mathbb{Q}^3] + \frac{c}{4}|\mathbb{Q}|^4$$

with constants $a, b, c \in \mathbb{R}$ which are determined by the material and temperature. More general choices of $F(\mathbb{Q})$ are considered in [4, 12, 23]. The coefficient $\xi \in \mathbb{R}$ measures the ratio between the rotation and aligning effects that a shear flow exerts over the directors. We assume $\xi = 0$ for simplicity. Thus, (1.2) and (1.3) become

$$(1.5) \quad \Sigma(\mathbb{Q}) = \Delta \mathbb{Q} \mathbb{Q} - \mathbb{Q} \Delta \mathbb{Q} - \nabla \mathbb{Q} \otimes \nabla \mathbb{Q}, \quad \mathbb{S}(\nabla u, \mathbb{Q}) = \Omega(u) \mathbb{Q} - \mathbb{Q} \Omega(u)$$

after using

$$\mathbb{Q} \mathcal{L}[\partial F(\mathbb{Q})] - \mathcal{L}[\partial F(\mathbb{Q})] \mathbb{Q} = \mathbb{Q} \partial F(\mathbb{Q}) - \partial F(\mathbb{Q}) \mathbb{Q} = 0.$$

We expect the same result shall hold for the general case $\xi \neq 0$, which will be considered in a future work. For more discussions regarding the physics of the model, we direct the readers to [6] and the references therein.

Regarding the mathematical aspects of the model, we briefly mention a few fundamental and relevant results in the literature without the intention to be complete. The existence of weak solutions to (1.1) was established by Paicu and Zarnescu [20, 19] in both two and three dimensions (3D), for $\xi = 0$ and small $|\xi| > 0$ respectively. Moreover, in 2D, the authors obtained global regular solutions. On a bounded domain in 3D, Abels et al. [1, 2] proved the existence and uniqueness of local strong solution subject to various boundary conditions. Regarding the long time behavior, Dai et al. [12] established the optimal decay rate for *weak* solutions in 3D, while optimal decay rates are usually obtained for regular solutions for other liquid crystal models (see [13, 14, 24]). Another interesting work is by Cavaterra et al. [7] who showed the existence of global strong solutions and decay rate of strong solutions in 2D with general $\xi \in \mathbb{R}$.

In this paper, we study the global regularity problem for the Q-tensor model (1.1) in 3D. The existence of global regular solutions for the 3D Navier-Stokes equation (NSE) is an outstanding open problem, and thus an open problem for (1.1) as well. For the 3D NSE, it is well known that the Prodi-Serrin condition

$$(1.6) \quad u \in L^l(0, \infty; L^r), \quad \frac{2}{l} + \frac{3}{r} = 1, \quad r > 3$$

or the Beale-Kato-Majda (BKM) condition (see [5])

$$(1.7) \quad \int_0^T \|\nabla \times u\|_\infty dt < \infty,$$

guarantees global regularity. For the inviscid NSE (Euler equation), Planchon [21] gave an extended BKM condition for regularity, namely

$$(1.8) \quad \limsup_{\epsilon \rightarrow 0} \sup_q \int_{T-\epsilon}^T \|\Delta_q(\nabla \times u)\|_\infty dt < c,$$

for a small enough constant c , where Δ_q denotes the Littlewood-Paley projection (see Section 2). Another improvement of the BKM criterion is given by Cheskidov and Shyvdokoy [9], as

$$(1.9) \quad \int_0^T \|\nabla \times u_{\leq Q}\|_{B_{\infty,\infty}^0} dt < \infty, \quad \text{for some wavenumber} \quad \Lambda(t) = 2^{Q(t)},$$

where $u_{\leq Q}$ denotes the low modes part of the velocity (see Section 2). Condition (1.9) is also weaker than all the Prodi-Serrin criteria (1.6). A wavenumber splitting method was introduced in [9] to achieve the goal. Recently, Cheskidov and Dai [8] refined the method and established the following regularity criterion

$$(1.10) \quad \limsup_{q \rightarrow \infty} \int_{T/2}^T 1_{q \leq Q(\tau)} \lambda_q \|u_q\|_\infty d\tau \leq c,$$

with a small constant c , and certain wavenumber $\Lambda(t) = 2^{Q(t)}$. This condition is weaker than all the conditions above, (1.6), (1.7), (1.8) and (1.9). Back to the Q-tensor model, it seems that not much previous work on this topic can be found in the literature, though Guillén-González and Rodríguez-Bellido pointed out that the classical Prodi-Serrin regularity criteria are not valid for the model due to the presence of the stretching terms (c.f. Remark 4, [17]). In another work of the same authors [16], they studied the so-called *weak regularity* for $(\partial_t u, \partial_t \mathbb{Q})$ which is different from the standard regularity problem as in the Navier-Stokes equation framework. They actually need to impose conditions on both of the velocity and the Q-tensor to imply the global in time of *weak regularity*. Regarding the standard (strong) regularity, they showed that with an additional assumption on the gradient of velocity as

$$\nabla u \in L^p(0, T; L^q), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 2 \leq q \leq 3,$$

the solution on $(0, T)$ does not blow-up at time T . In this paper, we shall apply the wavenumber splitting method to the Q-tensor model and obtain that a condition analogous to (1.9) or (1.10) solely imposed on velocity yields global regularity of (1.1) in 3D. Thus, automatically, the classical Prodi-Serrin or BKM condition solely on velocity is sufficient to imply global regularity for the model.

Based on the scaling of the flow equation in (1.1), we define the dissipation wavenumber $\Lambda(t)$ for the velocity as

$$(1.11) \quad \Lambda(t) = \min \left\{ \lambda_q : \lambda_p^{-1+\frac{3}{r}} \|u_p(t)\|_r < c_r \min\{\nu, \mu\}, \forall p > q, q \in \mathbb{N} \right\},$$

with $\lambda_q = 2^q$. For more notations, $u_p = \Delta_p u$ denotes the Littlewood-Paley projection of u , and c_r is a dimensionless constant that depends only on $r \in [2, 6)$. Take $Q(t) \in \mathbb{N}$ such that $\lambda_{Q(t)} = \Lambda(t)$. Our main results are stated below.

Theorem 1.1. *Let (u, \mathbb{Q}) be a weak solution to (1.1) on $[0, T]$. Assume that*

$$(1.12) \quad \int_0^T \|\nabla u_{\leq Q}(t)\|_{B_{\infty,\infty}^0} dt \leq \infty.$$

Then $(u(t), \mathbb{Q}(t))$ is regular on $(0, T]$.

Theorem 1.2. *[A stronger statement.] Let (u, \mathbb{Q}) be a weak solution to (1.1) on $[0, T]$. Assume that $(u(t), \mathbb{Q}(t))$ is regular on $(0, T)$, and*

$$(1.13) \quad \limsup_{q \rightarrow \infty} \int_{T/2}^T 1_{q \leq Q(\tau)} \lambda_q \|u_q\|_\infty d\tau \leq c,$$

for a small constant c . Then $(u(t), \mathbb{Q}(t))$ is regular on $(0, T]$.

Corollary 1.3. [Prodi-Serrin and BKM type criteria.] Let (u, \mathbb{Q}) be a weak solution to (1.1) as in Theorem 1.1. Assume that one of the following holds

$$(1.14) \quad u \in L^s(0, T; L^r(\mathbb{R}^3)), \quad \text{with } \frac{2}{s} + \frac{3}{r} = 1, \quad 3 < r < 6;$$

$$(1.15) \quad \int_0^T \|\nabla \times u(t)\|_\infty dt \leq \infty.$$

Then $(u(t), \mathbb{Q}(t))$ is regular on $(0, T]$.

The results have certainly a novelty value. First of all, each condition is solely imposed on the low frequency part of the velocity despite the fact that (1.1) is a coupled system of the NSE and the evolution of the \mathbb{Q} -tensor. Second, even though strong nonlinearity appears in the system, for instance the term $\nabla \cdot (\mathbb{H}\mathbb{Q} - \mathbb{Q}\mathbb{H})$, we are able to deal with it by revealing cancelations. More over, as stated in Corollary 1.3, the classical Prodi-Serrin and BKM type criteria are valid for the \mathbb{Q} -tensor system, which was previously unknown. Although, we would like to point out that there is a restriction on the index $r < 6$ for the Prodi-Serrin type criteria obtained in Corollary 1.3. Extending it to the full range $r \in (3, \infty]$ will be addressed in a future work.

We mention that the wavenumber splitting method has also been successfully applied to other dissipative equations, for instance, the supercritical surface quasi-geostrophic equation in [10], the magneto-hydrodynamics system in [8], and the Hall magneto-hydrodynamics system in [11].

The rest of the paper is planned as follows. In Section 2 we introduce some notations, recall the Littlewood-Paley theory, describe the energy laws of solutions to (1.1), and establish some commutator estimates. Section 3 is devoted to proving Theorem 1.1. The proof of Theorem 1.2 will be omitted since an identical analysis based on the proof of Theorem 1.1 can be found in previous work [8]. On the other hand, (1.15) obviously implies (1.12); and it is shown in [8] that (1.14) implies (1.12) as well. Thus, the proof of Corollary 1.3 will be not present either.

2. PRELIMINARIES

2.1. Notation. We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some absolute constant C , and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with some absolute constants C_1, C_2 . We write $\|\cdot\|_p = \|\cdot\|_{L^p}$. The symbol (\cdot, \cdot) stands for the L^2 -inner product.

Let S_0^d be the space of \mathbb{Q} -tensors in dimension d , namely,

$$S_0^d = \{\mathbb{Q} \in \mathbb{M}^{d \times d} : \mathbb{Q}_{ij} = \mathbb{Q}_{ji}, \text{tr}(\mathbb{Q}) = 0, i, j = 1, \dots, d\}.$$

We adapt the Frobenius norm of a matrix $|\mathbb{Q}| = \sqrt{\text{tr} \mathbb{Q}^2} = \sqrt{\mathbb{Q}_{\alpha\beta} \mathbb{Q}_{\alpha\beta}}$ and define Sobolev spaces of \mathbb{Q} -tensors in terms of this norm.

2.2. Littlewood-Paley decomposition. Our method relies on the Littlewood-Paley decomposition, which we briefly recall here. For a more detailed description on this theory, we direct the readers to the books [3] and [15].

We denote the Fourier transform and inverse Fourier transform by \mathcal{F} and \mathcal{F}^{-1} , respectively. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative radial function such that

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{3}{4} \\ 0, & \text{for } |\xi| \geq 1. \end{cases}$$

More bump functions are chosen as

$$\varphi(\xi) = \chi(\xi/2) - \chi(\xi), \quad \varphi_q(\xi) = \begin{cases} \varphi(\lambda_q^{-1}\xi) & \text{for } q \geq 0, \\ \chi(\xi) & \text{for } q = -1, \end{cases}$$

with $\lambda_q = 2^q$. Note that the sequence of the smooth functions $\{\varphi_q\}_{q \geq -1}$ forms a dyadic partition of unity. For a tempered distribution vector field u we define the Littlewood-Paley decomposition

$$\begin{cases} h = \mathcal{F}^{-1}\varphi, & \tilde{h} = \mathcal{F}^{-1}\chi, \\ u_q := \Delta_q u = \mathcal{F}^{-1}(\varphi(\lambda_q^{-1}\xi)\mathcal{F}u) = \lambda_q^n \int h(\lambda_q y) u(x-y) dy, & \text{for } q \geq 0, \\ u_{-1} = \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}u) = \int \tilde{h}(y) u(x-y) dy. \end{cases}$$

Then

$$u = \sum_{q=-1}^{\infty} u_q$$

holds in the distributional sense. To simplify the notation, we denote

$$\tilde{u}_q = u_{q-1} + u_q + u_{q+1}, \quad u_{\leq Q} = \sum_{q=-1}^Q u_q, \quad u_{(P,Q]} = \sum_{q=P+1}^Q u_q.$$

The Besov space $B_{p,\infty}^s$ is defined as follows.

Definition 2.1. Let $s \in \mathbb{R}$, and $1 \leq p \leq \infty$. The Besov space $B_{p,\infty}^s$ is the space of tempered distributions u such that the following norm

$$\|u\|_{B_{p,\infty}^s} = \sup_{q \geq -1} \lambda_q^s \|u_q\|_p$$

is finite.

Note that

$$\|u\|_{H^s} \sim \left(\sum_{q=-1}^{\infty} \lambda_q^{2s} \|u_q\|_2^2 \right)^{1/2},$$

for each $u \in H^s$ and $s \in \mathbb{R}$.

We recall Bernstein's inequality (c.f. [18]).

Lemma 2.2. Let n be the space dimension and $r \geq s \geq 1$. Then for all tempered distributions u ,

$$\|u_q\|_r \lesssim \lambda_q^{n(\frac{1}{s}-\frac{1}{r})} \|u_q\|_s.$$

2.3. Energy law, weak and strong solutions, maximum principle. Denote the energy functional as

$$E = \frac{1}{2}|u|^2 + \frac{1}{2}|\nabla \mathbb{Q}|^2 + F(\mathbb{Q}).$$

The basic energy law is given by (c.f. [20])

$$\frac{d}{dt} \int_{\mathbb{R}^3} E \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 + |\Delta \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})]|^2 \, dx = 0$$

provided that u and \mathbb{Q} vanish for large $|x|$.

We recall the standard definitions of weak and regular solutions to a differential equation system (see, e.g., [22]).

Definition 2.3. A weak solution of (1.1) on $[0, T]$ is a pair of functions (u, \mathbb{Q}) in the class

$$\begin{aligned} u &\in C_w([0, T]; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)), \\ \mathbb{Q} &\in C_w([0, T]; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)), \end{aligned}$$

satisfying

$$\begin{aligned} &(u(t), \varphi(t)) - (u_0, \varphi(0)) - \int_0^t (u(s), \partial_s \varphi(s)) + \nu(u(s), \Delta \phi(s)) \, ds \\ &= \int_0^t (u(s) \cdot \nabla \varphi(s), u(s)) + ([\mathbb{Q}(s)\mathbb{H}(s) - \mathbb{H}(s)\mathbb{Q}(s)] + \nabla \mathbb{Q}(s) \otimes \nabla \mathbb{Q}(s), \nabla \varphi(s)) \, ds, \\ &(\mathbb{Q}(t), \psi(t)) - (\mathbb{Q}_0, \psi(0)) - \int_0^t (\mathbb{Q}(s), \partial_s \psi(s)) + \mu(\mathbb{Q}(s), \Delta \psi(s)) \, ds \\ &= \int_0^t (u(s) \cdot \nabla \psi(s), \mathbb{Q}(s)) - (\Omega(u(s))\mathbb{Q}(s) - \mathbb{Q}(s)\Omega(u(s)), \psi(s)) \, ds, \end{aligned}$$

for all smooth functions $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^3; \mathbb{R}^3)$ with $\nabla_x \cdot \varphi = 0$, and $\psi \in C_0^\infty([0, T] \times \mathbb{R}^3; S_0^3)$.

Definition 2.4. A weak solution (u, \mathbb{Q}) of (1.1) is regular on a time interval \mathcal{I} if $\|u(t)\|_{H^s}$ and $\|\nabla \mathbb{Q}(t)\|_{H^s}$ are continuous on \mathcal{I} for some $s > \frac{1}{2}$.

In [12], we proved the following maximum principle for the \mathbb{Q} -tensor equation.

Lemma 2.5. Let (u, \mathbb{Q}) be a weak solution to (1.1) with initial data (u_0, \mathbb{Q}_0) . Then, for all $t > 0$, it holds

$$\|\mathbb{Q}(\cdot, t)\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq \|\mathbb{Q}_0\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}.$$

2.4. Commutators. To deal with the Littlewood-Paley projection of a product term, we often decompose it by the so-called Bony's paraproduct according to different types of interactions. Namely, for instance, we write

$$\begin{aligned} \Delta_q(u \cdot \nabla v) &= \sum_{|q-p| \leq 2} \Delta_q(u_{\leq p-2} \cdot \nabla v_p) + \sum_{|q-p| \leq 2} \Delta_q(u_p \cdot \nabla v_{\leq p-2}) \\ &\quad + \sum_{p \geq q-2} \Delta_q(\tilde{u}_p \cdot \nabla v_p). \end{aligned}$$

To reveal cancellations in the estimates, we also introduce the commutator

$$[\Delta_q, u_{\leq p-2} \cdot \nabla] v_p = \Delta_q(u_{\leq p-2} \cdot \nabla v_p) - u_{\leq p-2} \cdot \nabla \Delta_q v_p.$$

The following estimate holds.

Lemma 2.6. *Let u be a function with $\nabla \cdot u = 0$. For any $1 \leq r_1, r_2, r_3 \leq \infty$ satisfying $\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1}$, we have*

$$\|[\Delta_q, u_{\leq p-2} \cdot \nabla] v_q\|_{r_1} \lesssim \|v_q\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \|u_{p'}\|_{r_2}.$$

Proof: Following the definition of Δ_q , we infer

$$\begin{aligned} [\Delta_q, u_{\leq p-2} \cdot \nabla] v_q &= \int_{\mathbb{R}^3} \lambda_q^3 h(\lambda_q(x-y)) (u_{\leq p-2}(y) - u_{\leq p-2}(x)) \cdot \nabla v_q(y) dy \\ &= - \int_{\mathbb{R}^3} \lambda_q^3 \nabla h(\lambda_q(x-y)) (u_{\leq p-2}(y) - u_{\leq p-2}(x)) \otimes v_q(y) dy, \end{aligned}$$

thanks to the fact $\nabla \cdot u_{\leq p-2} = 0$. Thus, by Young's inequality,

$$\begin{aligned} \|[\Delta_q, u_{\leq p-2} \cdot \nabla] u_q\|_{r_1} &\lesssim \|v_q\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \|u_{p'}\|_{r_2} \left| \int_{\mathbb{R}^3} \lambda_q^3 |x-y| \nabla h(\lambda_q(x-y)) dy \right| \\ &\lesssim \|v_q\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \|u_{p'}\|_{r_2}, \end{aligned}$$

for $\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1}$. □

One can see another benefit of using commutator is that the derivative on high modes can be moved onto the low modes. We define a few more commutators regarding the \mathbb{Q} tensor terms in the same spirit, as follows

$$\begin{aligned} [\Delta_q, \Delta \mathbb{Q}_p] \mathbb{Q}_{\leq p-2} &= \Delta_q (\Delta \mathbb{Q}_p \mathbb{Q}_{\leq p-2}) - \Delta \Delta_q \mathbb{Q}_p \mathbb{Q}_{\leq p-2}, \\ [\Delta_q, \mathbb{Q}_{\leq p-2} \Delta] \mathbb{Q}_p &= \Delta_q (\mathbb{Q}_{\leq p-2} \Delta \mathbb{Q}_p) - \mathbb{Q}_{\leq p-2} \Delta \Delta_q \mathbb{Q}_p, \\ [\Delta_q, \Omega(u)_p] \mathbb{Q}_{\leq p-2} &= \Delta_q (\Omega(u)_p \mathbb{Q}_{\leq p-2}) - \Delta_q \Omega(u)_p \mathbb{Q}_{\leq p-2}, \\ [\Delta_q, \mathbb{Q}_{\leq p-2}] \Omega(u)_p &= \Delta_q (\mathbb{Q}_{\leq p-2} \Omega(u)_p) - \mathbb{Q}_{\leq p-2} \Delta_q \Omega(u)_p, \\ [\Delta_q, \nabla \mathbb{Q}_p] \nabla \mathbb{Q}_{\leq p-2} &= \Delta_q (\nabla \mathbb{Q}_p \otimes \nabla \mathbb{Q}_{\leq p-2}) - \nabla \Delta_q \mathbb{Q}_p \otimes \nabla \mathbb{Q}_{\leq p-2}, \\ [\Delta_q, \nabla \mathbb{Q}_{\leq p-2}] \nabla \mathbb{Q}_p &= \Delta_q (\nabla \mathbb{Q}_{\leq p-2} \otimes \nabla \mathbb{Q}_p) - \nabla \mathbb{Q}_{\leq p-2} \otimes \nabla \Delta_q \mathbb{Q}_p. \end{aligned}$$

Lemma 2.7. *For any $1 \leq r_1, r_2, r_3 \leq \infty$ satisfying $\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1}$, we have*

$$\begin{aligned} \|[\Delta_q, \Delta \mathbb{Q}_p] \mathbb{Q}_{\leq p-2}\|_{r_1} &\lesssim \|\mathbb{Q}_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{r_2}, \\ \|[\Delta_q, \mathbb{Q}_{\leq p-2} \Delta] \mathbb{Q}_p\|_{r_1} &\lesssim \|\mathbb{Q}_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{r_2}, \\ \|[\Delta_q, \Omega(u)_p] \mathbb{Q}_{\leq p-2}\|_{r_1} &\lesssim \|u_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \|\mathbb{Q}_{p'}\|_{r_2}, \\ \|[\Delta_q, \mathbb{Q}_{\leq p-2}] \Omega(u)_p\|_{r_1} &\lesssim \|u_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \|\mathbb{Q}_{p'}\|_{r_2}, \\ \|[\Delta_q, \nabla \mathbb{Q}_p] \nabla \mathbb{Q}_{\leq p-2}\|_{r_1} &\lesssim \|\mathbb{Q}_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{r_2}, \\ \|[\Delta_q, \nabla \mathbb{Q}_{\leq p-2}] \nabla \mathbb{Q}_p\|_{r_1} &\lesssim \|\mathbb{Q}_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{r_2}. \end{aligned}$$

Proof: Only the first inequality will be proven in the following, and other ones can be obtained in an analogous way. Again, following the definition of Δ_q and applying integration by parts, the commutator can be written as

$$\begin{aligned} [\Delta_q, \Delta \mathbb{Q}_p] \mathbb{Q}_{\leq p-2} &= \int_{\mathbb{R}^3} \lambda_q^3 h(\lambda_q(x-y)) \Delta \mathbb{Q}_p(y) (\mathbb{Q}_{\leq p-2}(y) - \mathbb{Q}_{\leq p-2}(x)) dy \\ &= \int_{\mathbb{R}^3} \lambda_q^3 \Delta h(\lambda_q(x-y)) \mathbb{Q}_p(y) (\mathbb{Q}_{\leq p-2}(y) - \mathbb{Q}_{\leq p-2}(x)) dy \\ &\quad + 2 \int_{\mathbb{R}^3} \lambda_q^3 \nabla h(\lambda_q(x-y)) \mathbb{Q}_p(y) \nabla (\mathbb{Q}_{\leq p-2}(y) - \mathbb{Q}_{\leq p-2}(x)) dy \\ &\quad + \int_{\mathbb{R}^3} \lambda_q^3 h(\lambda_q(x-y)) \mathbb{Q}_p(y) \Delta (\mathbb{Q}_{\leq p-2}(y) - \mathbb{Q}_{\leq p-2}(x)) dy. \end{aligned}$$

Thus, by Young's inequality, we infer

$$\begin{aligned} &\|[\Delta_q, \Delta \mathbb{Q}_p] \mathbb{Q}_{\leq p-2}\|_{r_1} \\ &\lesssim \|\mathbb{Q}_p\|_{r_3} \sum_{p' \leq p-2} \|\nabla^2 \mathbb{Q}_{p'}\|_{r_2} \left| \int_{\mathbb{R}^3} \lambda_q^3 |x-y|^2 \Delta h(\lambda_q(x-y)) dy \right| \\ &\quad + \|\mathbb{Q}_p\|_{r_3} \sum_{p' \leq p-2} \|\nabla^2 \mathbb{Q}_{p'}\|_{r_2} \left| \int_{\mathbb{R}^3} \lambda_q^3 |x-y| \nabla h(\lambda_q(x-y)) dy \right| \\ &\quad + \|\mathbb{Q}_p\|_{r_3} \sum_{p' \leq p-2} \|\Delta \mathbb{Q}_{p'}\|_{r_2} \left| \int_{\mathbb{R}^3} \lambda_q^3 h(\lambda_q(x-y)) dy \right| \\ &\lesssim \|\mathbb{Q}_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{r_2}, \end{aligned}$$

for $\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1}$.

□

Similar computation strategy yields the following estimates.

Lemma 2.8. *For any $1 \leq r_1, r_2, r_3 \leq \infty$ satisfying $\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1}$, we have*

$$\begin{aligned} \|\nabla([\Delta_q, \Omega(u)_p] \mathbb{Q}_{\leq p-2})\|_{r_1} &\lesssim \|u_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{r_2}, \\ \|\nabla([\Delta_q, \mathbb{Q}_{\leq p-2}] \Omega(u)_p)\|_{r_1} &\lesssim \|u_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{r_2}. \end{aligned}$$

3. REGULARITY CRITERION

In this section we will establish the regularity criterion in Theorem 1.1. Let $(u(t), \mathbb{Q}(t))$ be a weak solution of (1.1) on $[0, T]$. Based on the scaling of the Navier-Stokes equation, we define a dissipation wavenumber corresponding to velocity as

$$(3.16) \quad \Lambda(t) = \min \left\{ \lambda_q : \lambda_p^{-1+\frac{3}{r}} \|u_p(t)\|_r < c_r \min\{\nu, \mu\}, \forall p > q, q \in \mathbb{N} \right\},$$

where c_r is an adimensional constant that depends only on r , and $r \in [2, 6)$. We point out that the quantity $\lambda_p^{-1+\frac{3}{r}} \|u_p(t)\|_r$ is scaling invariant. Let $Q(t) \in \mathbb{N}$ be such that $\lambda_{Q(t)} = \Lambda(t)$. It follows immediately that

$$\|u_p(t)\|_r < \lambda_p^{1-\frac{3}{r}} c_r \min\{\nu, \mu\}, \quad \forall p > Q(t),$$

and

$$(3.17) \quad \|u_{Q(t)}(t)\|_r \geq c_r \min\{\nu, \mu\} \Lambda^{1-\frac{3}{r}}(t),$$

provided $1 < \Lambda(t) < \infty$. We also denote

$$f(t) = \|\nabla u_{\leq Q(t)}\|_{B_{\infty,\infty}^0}.$$

3.1. Proof of Theorem 1.1. In order to prove that (u, \mathbb{Q}) does not blow up at T , it is sufficient to show that $\|u(t)\|_{H^s} + \|\nabla \mathbb{Q}(t)\|_{H^s}$ is bounded on $(0, T)$ for some $s > \frac{1}{2}$. Multiplying equations (1.1) with $\Delta_q^2 u$ and $\Delta_q^2 \Delta \mathbb{Q}$ respectively yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_q\|_2^2 &\leq -\nu \|\nabla u_q\|_2^2 - \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla u) \cdot u_q \, dx - \int_{\mathbb{R}^3} \Delta_q(\Sigma(\mathbb{Q})) \cdot \nabla u_q \, dx, \\ \frac{1}{2} \frac{d}{dt} \|\nabla \mathbb{Q}_q\|_2^2 &\leq -\mu \|\Delta \mathbb{Q}_q\|_2^2 - \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla \mathbb{Q}) \cdot \Delta \mathbb{Q}_q \, dx \\ &\quad + \int_{\mathbb{R}^3} \Delta_q(\mathbb{S}(\nabla u, \mathbb{Q})) \cdot \Delta \mathbb{Q}_q \, dx - \int_{\mathbb{R}^3} \Delta_q \mathcal{L}[\partial F(\mathbb{Q})] \Delta \mathbb{Q}_q \, dx. \end{aligned}$$

Adding the above two inequalities, multiplying by λ_q^{2s} , and adding them for all $q \geq -1$, we obtain

$$(3.18) \quad \frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2s} (\|u_q\|_2^2 + \|\nabla \mathbb{Q}_q\|_2^2) \leq - \sum_{q \geq -1} \lambda_q^{2s} (\nu \|\nabla u_q\|_2^2 + \mu \|\Delta \mathbb{Q}_q\|_2^2) - (I + J + K + L + M),$$

with

$$\begin{aligned} I &= \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla u) \cdot u_q \, dx, & J &= - \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(\Sigma(\mathbb{Q})) \nabla u_q \, dx, \\ K &= \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla \mathbb{Q}) \cdot \Delta \mathbb{Q}_q \, dx, & L &= \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(\mathbb{S}(\nabla u, \mathbb{Q})) \cdot \Delta \mathbb{Q}_q \, dx, \\ M &= \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q \mathcal{L}[\partial F(\mathbb{Q})] \Delta \mathbb{Q}_q \, dx. \end{aligned}$$

Thanks to the maximum principle stated in Lemma 2.5,

$$\|\mathbb{Q}(t, \cdot)\|_{L^\infty} \leq C \quad \text{for all } t \geq 0,$$

the term M can be estimated immediately. Recalling $F(\mathbb{Q}) = \frac{a}{2} |\mathbb{Q}|^2 + \frac{b}{3} \text{tr}[\mathbb{Q}^3] + \frac{c}{4} |\mathbb{Q}|^4$, we have

$$\begin{aligned} |M| &\leq \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q \mathcal{L}[\partial F(\mathbb{Q})] \Delta \mathbb{Q}_q| \, dx \\ &\leq \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} |a \mathbb{Q}_q + b \Delta_q(\mathbb{Q}^2) + c \Delta_q(\mathbb{Q} \text{tr} \mathbb{Q}^2)| |\Delta \mathbb{Q}_q| \, dx \\ (3.19) \quad &\lesssim (1 + \|\mathbb{Q}\|_\infty + \|\mathbb{Q}\|_\infty^2) \sum_{q \geq -1} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2 \\ &\lesssim \sum_{q \geq -1} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2. \end{aligned}$$

Regarding the other terms, the main idea is to decompose them into high frequency and low frequency parts (by Q), such that the high frequency parts get

absorbed by the diffusion term $\nu \|u\|_{H^{s+1}}^2 + \mu \|\mathbb{Q}\|_{H^{s+2}}^2$. The term I can be dealt with the same way as for the Navier-Stokes equation in [8], and the estimate is

$$(3.20) \quad |I| \lesssim c_r \mu \sum_{q>Q} \lambda_q^{2s+2} \|u_q\|_2^2 + Q f(t) \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^2.$$

We proceed the estimate for J , L and K in the following. Recall

$$\Sigma(\mathbb{Q}) = \Delta \mathbb{Q} \mathbb{Q} - \mathbb{Q} \Delta \mathbb{Q} - \nabla \mathbb{Q} \otimes \nabla \mathbb{Q}, \quad \mathbb{S}(\nabla u, \mathbb{Q}) = \Omega(u) \mathbb{Q} - \mathbb{Q} \Omega(u).$$

It follows that

$$\begin{aligned} J &= - \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\Delta \mathbb{Q} \mathbb{Q} - \mathbb{Q} \Delta \mathbb{Q}) \nabla u_q \, dx + \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\nabla \mathbb{Q} \otimes \nabla \mathbb{Q}) \nabla u_q \, dx \\ &=: J_1 + J_2. \end{aligned}$$

We shall discover cancelations in $J_1 + L$ and $J_2 + K$ which are essential to obtain the ultimate estimate. Using Bony's paraproduct decomposition and the commutator notation, J_1 is decomposed as

$$\begin{aligned} J_1 &= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\Delta \mathbb{Q}_p \mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq p-2} \Delta \mathbb{Q}_p) \nabla u_q \, dx \\ &\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\Delta \mathbb{Q}_{\leq p-2} \mathbb{Q}_p - \mathbb{Q}_p \Delta \mathbb{Q}_{\leq p-2}) \nabla u_q \, dx \\ &\quad + \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\Delta \mathbb{Q}_p \tilde{\mathbb{Q}}_p - \mathbb{Q}_p \Delta \tilde{\mathbb{Q}}_p) \nabla u_q \, dx \\ &=: J_{11} + J_{12} + J_{13}, \end{aligned}$$

with

$$\begin{aligned} J_{11} &= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ([\Delta_q, \Delta \mathbb{Q}_p] \mathbb{Q}_{\leq p-2} - [\Delta_q, \mathbb{Q}_{\leq p-2} \Delta] \mathbb{Q}_p) \nabla u_q \, dx \\ &\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (\Delta \Delta_q \mathbb{Q}_p \mathbb{Q}_{\leq q-2} - \mathbb{Q}_{\leq q-2} \Delta \Delta_q \mathbb{Q}_p) \nabla u_q \, dx \\ &\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (\Delta \Delta_q \mathbb{Q}_p (\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}) - (\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}) \Delta \Delta_q \mathbb{Q}_p) \nabla u_q \, dx \\ &=: J_{111} + J_{112} + J_{113}. \end{aligned}$$

Similarly, L can be decomposed as,

$$\begin{aligned} L &= - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\Omega(u)_p \mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq p-2} \Omega(u)_p) \cdot \Delta \mathbb{Q}_q \, dx \\ &\quad - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\Omega(u)_{\leq p-2} \mathbb{Q}_p - \mathbb{Q}_p \Omega(u)_{\leq p-2}) \cdot \Delta \mathbb{Q}_q \, dx \\ &\quad - \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\Omega(u)_p \tilde{\mathbb{Q}}_p - \mathbb{Q}_p \Omega(\tilde{u})_p) \cdot \Delta \mathbb{Q}_q \, dx \\ &=: L_1 + L_2 + L_3, \end{aligned}$$

with

$$\begin{aligned}
L_1 &= - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ([\Delta_q, \Omega(u)_p] \mathbb{Q}_{\leq p-2} - [\Delta_q, \mathbb{Q}_{\leq p-2}] \Omega(u)_p) \cdot \Delta \mathbb{Q}_q \, dx \\
&\quad - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (\Delta_q \Omega(u)_p \mathbb{Q}_{\leq q-2} - \mathbb{Q}_{\leq q-2} \Delta_q \Omega(u)_p) \cdot \Delta \mathbb{Q}_q \, dx \\
&\quad - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (\Delta_q \Omega(u)_p (\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{q-2}) - (\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}) \Delta_q \Omega(u)_p) \cdot \Delta \mathbb{Q}_q \, dx \\
&= L_{11} + L_{12} + L_{13}.
\end{aligned}$$

Note that $\sum_{|q-p| \leq 2} \Delta \Delta_q \mathbb{Q}_p = \Delta \mathbb{Q}_q$ and $\sum_{|q-p| \leq 2} \Delta_q \Omega(u)_p = \Omega(u)_q$. Thus

$$\begin{aligned}
J_{112} + L_{12} &= \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (\Delta \mathbb{Q}_q \mathbb{Q}_{\leq q-2} - \mathbb{Q}_{\leq q-2} \Delta \mathbb{Q}_q) \nabla u_q \, dx \\
&\quad - \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (\Omega(u)_q \mathbb{Q}_{\leq q-2} - \mathbb{Q}_{\leq q-2} \Omega(u)_q) \cdot \Delta \mathbb{Q}_q \, dx \\
&= 0.
\end{aligned}$$

The other terms in J_1 and L are estimated as follows. We further split J_{111} into high and low frequency parts as

$$\begin{aligned}
|J_{111}| &\leq 2 \sum_{q > Q} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |[\Delta_q, \Delta \mathbb{Q}_p] \mathbb{Q}_{\leq p-2} \nabla u_q| \, dx \\
&\quad + 2 \sum_{-1 \leq q \leq Q} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |[\Delta_q, \Delta \mathbb{Q}_p] \mathbb{Q}_{\leq p-2} \nabla u_q| \, dx \\
&\equiv J_{1111} + J_{1112}.
\end{aligned}$$

By Hölder's inequality, the commutator estimate in Lemma 2.7, the definition of Λ_r , and Jensen's inequality, it follows that

$$\begin{aligned}
J_{1111} &\lesssim \sum_{q > Q} \lambda_q^{2s} \|\nabla u_q\|_r \sum_{|q-p| \leq 2} \|\mathbb{Q}_p\|_2 \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{\frac{2r}{r-2}} \\
&\lesssim c_r \mu \sum_{q > Q} \lambda_q^{2s+2-\frac{3}{r}} \sum_{|q-p| \leq 2} \|\mathbb{Q}_p\|_2 \sum_{p' \leq q} \lambda_{p'}^{2+\frac{3}{r}} \|\mathbb{Q}_{p'}\|_2 \\
&\lesssim c_r \mu \sum_{q > Q-2} \lambda_q^{2s+2-\frac{3}{r}} \|\mathbb{Q}_q\|_2 \sum_{p' \leq q} \lambda_{p'}^{2+\frac{3}{r}} \|\mathbb{Q}_{p'}\|_2 \\
&\lesssim c_r \mu \sum_{q > Q-2} \lambda_q^{s+2} \|\mathbb{Q}_q\|_2 \sum_{p' \leq q} \lambda_{p'}^{s+2} \|\mathbb{Q}_{p'}\|_2 \lambda_{q-p'}^{s-\frac{3}{r}} \\
&\lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2,
\end{aligned}$$

where we needed $2 \leq r < \frac{3}{s}$. Also, by Hölder's inequality, the definition of $f(t)$ and Jensen's inequality,

$$\begin{aligned}
J_{1112} &\lesssim \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \|\nabla u_q\|_\infty \sum_{|q-p| \leq 2} \|\mathbb{Q}_p\|_2 \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_2 \\
&\lesssim f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \sum_{|q-p| \leq 2} \|\mathbb{Q}_p\|_2 \sum_{p' \leq q} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_2 \\
&\lesssim f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{2s} \|\mathbb{Q}_q\|_2 \sum_{p' \leq q} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_2 \\
&\lesssim f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{s+1} \|\mathbb{Q}_q\|_2 \sum_{p' \leq q} \lambda_{p'}^{s+1} \|\mathbb{Q}_{p'}\|_2 \lambda_{p'-q}^{1-s} \\
&\lesssim f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2,
\end{aligned}$$

where we used $\frac{1}{2} < s < 1$. Similar analysis yields

$$\begin{aligned}
|J_{113}| &\leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (\Delta \Delta_q \mathbb{Q}_p (\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}) - (\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}) \Delta \Delta_q \mathbb{Q}_p) \nabla u_q \, dx \\
&\leq \sum_{q > Q} \sum_{|q-p| \leq 2} \lambda_q^{2s+3} \|u_q\|_r \|\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}\|_2 \|\mathbb{Q}_p\|_{\frac{2r}{r-2}} \\
&\quad + \sum_{-1 \leq q \leq Q} \sum_{|q-p| \leq 2} \lambda_q^{2s+3} \|u_q\|_\infty \|\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}\|_2 \|\mathbb{Q}_p\|_2 \\
&\lesssim c_r \mu \sum_{q > Q} \lambda_q^{2s+4-\frac{3}{r}} \sum_{|q-p| \leq 2} \lambda_p^{\frac{3}{r}} \|\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}\|_2 \|\mathbb{Q}_p\|_2 \\
&\quad + f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s+2} \sum_{|q-p| \leq 2} \|\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}\|_2 \|\mathbb{Q}_p\|_2 \\
&\lesssim c_r \mu \sum_{q > Q} \lambda_q^{2s+4-\frac{3}{r}} \sum_{q-3 \leq p \leq q+2} \lambda_p^{\frac{3}{r}} \|\mathbb{Q}_p\|_2^2 + f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2 \\
&\lesssim c_r \mu \sum_{q > Q-3} \lambda_q^{2s+4} \|\mathbb{Q}_p\|_2^2 + f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.
\end{aligned}$$

Notice that J_{12} enjoys the same estimate as J_{111} . While for J_{13} we have

$$\begin{aligned}
|J_{13}| &\lesssim \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \left| \Delta_q (\Delta \mathbb{Q}_p \tilde{\mathbb{Q}}_p - \mathbb{Q}_p \Delta \tilde{\mathbb{Q}}_p) \nabla u_q \right| \, dx \\
&\lesssim \sum_{q > Q} \lambda_q^{2s+1} \|u_q\|_\infty \sum_{p \geq q-2} \lambda_p^2 \|\mathbb{Q}_p\|_2^2 + \sum_{-1 \leq q \leq Q} \lambda_q^{2s+1} \|u_q\|_\infty \sum_{p \geq q-2} \lambda_p^2 \|\mathbb{Q}_p\|_2^2 \\
&\lesssim \sum_{q > Q} \lambda_q^{2s+1+\frac{3}{r}} \|u_q\|_r \sum_{p \geq q-2} \lambda_p^2 \|\mathbb{Q}_p\|_2^2 + f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \sum_{p \geq q-2} \lambda_p^2 \|\mathbb{Q}_p\|_2^2 \\
&\lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+4} \|\mathbb{Q}_p\|_2^2 \sum_{Q < q \leq p+2} \lambda_{q-p}^{2s+2} + f(t) \sum_{p \geq -1} \lambda_p^{2s+2} \|\mathbb{Q}_p\|_2^2 \sum_{-1 \leq q \leq p} \lambda_{q-p}^{2s} \\
&\lesssim c_r \mu \sum_{q > Q} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.
\end{aligned}$$

Therefore, for $2 \leq r < \frac{3}{s}$ and $\frac{1}{2} < s < 1$,

$$(3.21) \quad |J_1| \lesssim c_r \mu \sum_{q>Q-3} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.$$

Applying integration by parts to L_{11} yields

$$\begin{aligned} L_{11} &= - \sum_{p \leq Q} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \nabla ([\Delta_q, \Omega(u)_p] \mathbb{Q}_{\leq p-2} - [\Delta_q, \mathbb{Q}_{\leq p-2}] \Omega(u)_p) \nabla \mathbb{Q}_q \, dx \\ &\quad - \sum_{p > Q} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \nabla ([\Delta_q, \Omega(u)_p] \mathbb{Q}_{\leq p-2} - [\Delta_q, \mathbb{Q}_{\leq p-2}] \Omega(u)_p) \nabla \mathbb{Q}_q \, dx \\ &\equiv L_{111} + L_{112}. \end{aligned}$$

By Lemma 2.8, Young's inequality and Jensen's inequality, we infer

$$\begin{aligned} |L_{111}| &\lesssim \sum_{p \leq Q} \sum_{|q-p| \leq 2} \lambda_p^{2s} \|u_p\|_\infty \|\nabla \mathbb{Q}_q\|_2 \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_2 \\ &\lesssim f(t) \sum_{p \leq Q} \lambda_p^{2s} \|\mathbb{Q}_p\|_2 \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_2 \\ &\lesssim f(t) \sum_{p \leq Q} \lambda_p^{s+1} \|\mathbb{Q}_p\|_2 \sum_{p' \leq p-2} \lambda_{p'}^{s+1} \|\mathbb{Q}_{p'}\|_2 \lambda_{p-p'}^{s-1} \\ &\lesssim f(t) \sum_{p \leq Q} \lambda_p^{2s+2} \|\mathbb{Q}_p\|_2^2 \end{aligned}$$

for $s < 1$; while

$$\begin{aligned} |L_{112}| &\lesssim \sum_{p > Q} \sum_{|q-p| \leq 2} \lambda_p^{2s} \|u_p\|_r \|\nabla \mathbb{Q}_q\|_2 \sum_{p' \leq p-2} \lambda_{p'}^2 \|\mathbb{Q}_{p'}\|_{\frac{2r}{r-2}} \\ &\lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+2-\frac{3}{r}} \|\mathbb{Q}_p\|_2 \sum_{p' \leq p-2} \lambda_{p'}^{2+\frac{3}{r}} \|\mathbb{Q}_{p'}\|_2 \\ &\lesssim c_r \mu \sum_{p > Q} \lambda_p^{s+2} \|\mathbb{Q}_p\|_2 \sum_{p' \leq p-2} \lambda_{p'}^{s+2} \|\mathbb{Q}_{p'}\|_2 \lambda_{p'-p}^{\frac{3}{r}-s} \\ &\lesssim c_r \mu \sum_{p \geq -1} \lambda_p^{2s+4} \|\mathbb{Q}_p\|_2^2 \end{aligned}$$

for $s < \frac{3}{r}$.

Notice that L_{13} can be estimated similarly to J_{113} and L_3 can be estimated as J_{13} . Thus

$$|L_{13}| + |L_3| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.$$

To estimate L_2 , we split the summation first as

$$\begin{aligned}
|L_2| &\leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q(\Omega(u)_{\leq p-2} \mathbb{Q}_p - \mathbb{Q}_p \Omega(u)_{\leq p-2}) \cdot \Delta \mathbb{Q}_q| dx \\
&\lesssim \sum_{-1 \leq p \leq Q+2} \sum_{|p-q| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q(\Omega(u)_{\leq p-2} \mathbb{Q}_p) \Delta \mathbb{Q}_q| dx \\
&\quad + \sum_{p > Q+2} \sum_{|p-q| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q(\Omega(u)_{\leq Q} \mathbb{Q}_p) \Delta \mathbb{Q}_q| dx \\
&\quad + \sum_{p > Q+2} \sum_{|p-q| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q(\Omega(u)_{(Q, p-2]} \mathbb{Q}_p) \Delta \mathbb{Q}_q| dx \\
&\equiv L_{21} + L_{22} + L_{23}.
\end{aligned}$$

Then using Hölder's inequality and the definition of $f(t)$ we obtain

$$\begin{aligned}
L_{21} &\lesssim \sum_{-1 \leq p \leq Q+2} \|\nabla u_{\leq p-2}\|_{\infty} \|\mathbb{Q}_p\|_2 \sum_{|p-q| \leq 2} \lambda_q^{2s} \|\Delta \mathbb{Q}_q\|_2 \\
&\lesssim Q f(t) \sum_{-1 \leq p \leq Q+2} \|\mathbb{Q}_p\|_2 \sum_{|p-q| \leq 2} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2 \\
&\lesssim Q f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
L_{22} &\lesssim \sum_{p > Q+2} \|\nabla u_{\leq Q}\|_{\infty} \|\mathbb{Q}_p\|_2 \sum_{|p-q| \leq 2} \lambda_q^{2s} \|\Delta \mathbb{Q}_q\|_2 \\
&\lesssim Q f(t) \sum_{p > Q+2} \|\mathbb{Q}_p\|_2 \sum_{|p-q| \leq 2} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2 \\
&\lesssim Q f(t) \sum_{q > Q} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.
\end{aligned}$$

Applying Hölder's inequality, the definition of Λ_r and Jensen's inequality yields

$$\begin{aligned}
L_{23} &\lesssim \sum_{p > Q+2} \|\nabla u_{(Q, p-2]}\|_r \|\mathbb{Q}_p\|_{\frac{2r}{r-2}} \sum_{|p-q| \leq 2} \lambda_q^{2s} \|\Delta \mathbb{Q}_q\|_2 \\
&\lesssim \sum_{p > Q+2} \lambda_p^{\frac{3}{r}} \|\mathbb{Q}_p\|_2 \sum_{|p-q| \leq 2} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2 \sum_{Q < p' \leq p-2} \lambda_{p'} \|u_{p'}\|_r \\
&\lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+2+\frac{3}{r}} \|\mathbb{Q}_p\|_2^2 \sum_{Q < p' \leq p-2} \lambda_{p'}^{2-\frac{3}{r}} \\
&\lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+4} \|\mathbb{Q}_p\|_2^2 \sum_{Q < p' \leq p-2} \lambda_{p'-p}^{2-\frac{3}{r}} \\
&\lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+4} \|\mathbb{Q}_p\|_2^2,
\end{aligned}$$

since $r \geq 2$. Thus we have established that

$$(3.22) \quad |L| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2 + Q f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.$$

We deal with $J_2 + K$ now. As before, J_2 can be decomposed as

$$\begin{aligned}
J_2 &= \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\nabla \mathbb{Q} \otimes \nabla \mathbb{Q}) \nabla u_q dx \\
&= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\nabla \mathbb{Q}_{\leq p-2} \otimes \nabla \mathbb{Q}_p) \nabla u_q dx \\
&\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\nabla \mathbb{Q}_p \otimes \nabla \mathbb{Q}_{\leq p-2}) \nabla u_q dx \\
&\quad + \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\nabla \mathbb{Q}_p \otimes \nabla \tilde{\mathbb{Q}}_p) \nabla u_q dx \\
&= J_{21} + J_{22} + J_{23}.
\end{aligned}$$

Applying the commutator notation, we can rewrite J_{21} and J_{22}

$$\begin{aligned}
J_{21} &= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, \nabla \mathbb{Q}_{\leq p-2}] \nabla \mathbb{Q}_p \nabla u_q dx \\
&\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \nabla \mathbb{Q}_{\leq q-2} \otimes \nabla \Delta_q \mathbb{Q}_p \nabla u_q dx \\
&\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (\nabla \mathbb{Q}_{\leq p-2} - \nabla \mathbb{Q}_{\leq q-2}) \otimes \nabla \Delta_q \mathbb{Q}_p \nabla u_q dx \\
&\equiv J_{211} + J_{212} + J_{213};
\end{aligned}$$

and

$$\begin{aligned}
J_{22} &= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, \nabla \mathbb{Q}_p] \nabla \mathbb{Q}_{\leq p-2} \nabla u_q dx \\
&\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \nabla \Delta_q \mathbb{Q}_p \otimes \nabla \mathbb{Q}_{\leq q-2} \nabla u_q dx \\
&\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \nabla \Delta_q \mathbb{Q}_p \otimes (\nabla \mathbb{Q}_{\leq p-2} - \nabla \mathbb{Q}_{\leq q-2}) \nabla u_q dx \\
&\equiv J_{221} + J_{222} + J_{223}.
\end{aligned}$$

Since $\sum_{|p-q| \leq 2} \nabla \Delta_q \mathbb{Q}_p = \nabla \mathbb{Q}_q$, we have

$$J_{212} + J_{222} = \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \nabla \mathbb{Q}_{\leq q-2} \otimes \nabla \mathbb{Q}_q \nabla u_q dx + \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_{\leq q-2} \nabla u_q dx$$

which will be estimated together with K_{22} later.

We also decompose K by Bony's paraproduct,

$$\begin{aligned}
K &= \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla \mathbb{Q}) \cdot \Delta \mathbb{Q}_q \, dx \\
&= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_{\leq p-2} \cdot \nabla \mathbb{Q}_p) \cdot \Delta \mathbb{Q}_q \, dx \\
&\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla \mathbb{Q}_{\leq p-2}) \cdot \Delta \mathbb{Q}_q \, dx \\
&\quad + \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla \tilde{\mathbb{Q}}_p) \cdot \Delta \mathbb{Q}_q \, dx \\
&= K_1 + K_2 + K_3,
\end{aligned}$$

with

$$\begin{aligned}
K_1 &= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, u_{\leq p-2} \cdot \nabla] \mathbb{Q}_p \cdot \Delta \mathbb{Q}_q \, dx \\
&\quad + \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (u_{\leq q-2} \cdot \nabla) \mathbb{Q}_q \cdot \Delta \mathbb{Q}_q \, dx \\
&\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla \Delta_q \mathbb{Q}_p \cdot \Delta \mathbb{Q}_q \, dx \\
&= K_{11} + K_{12} + K_{13};
\end{aligned}$$

and

$$\begin{aligned}
K_2 &= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla \mathbb{Q}_{\leq p-2}) \cdot \Delta \mathbb{Q}_q \, dx \\
&= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, u_p \cdot \nabla] \mathbb{Q}_{\leq p-2} \cdot \Delta \mathbb{Q}_q \, dx \\
&\quad + \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (u_q \cdot \nabla) \mathbb{Q}_{\leq q-2} \cdot \Delta \mathbb{Q}_q \, dx \\
&\quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (\Delta_q u_p \cdot \nabla) (\mathbb{Q}_{\leq p-2} - \mathbb{Q}_{\leq q-2}) \cdot \Delta \mathbb{Q}_q \, dx \\
&\equiv K_{21} + K_{22} + K_{23}.
\end{aligned}$$

Here we used $\sum_{|q-p| \leq 2} \nabla \Delta_p \mathbb{Q}_q = \mathbb{Q}_q$ and $\sum_{|p-q| \leq 2} \Delta_q u_p = u_q$ to obtain K_{12} and K_{22} , respectively.

We claim that

$$(3.23) \quad J_{212} + J_{222} + K_{22} = - \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (u_q \cdot \nabla) \mathbb{Q}_q \Delta \mathbb{Q}_{\leq q-2} \, dx.$$

Indeed, denote $\mathbb{A} = \mathbb{Q}_{\leq q-2}$ and $\mathbb{B} = \mathbb{Q}_q$. Applying integration by parts and $\nabla \cdot u_q = 0$ yields

$$\begin{aligned}
& \int_{\mathbb{R}^3} (\nabla \mathbb{Q}_{\leq q-2} \otimes \nabla \mathbb{Q}_q) \nabla u_q dx + \int_{\mathbb{R}^3} (\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_{\leq q-2}) \nabla u_q dx \\
&= \int_{\mathbb{R}^3} \mathbb{A}_{\gamma\delta,\alpha} \mathbb{B}_{\gamma\delta,\beta} (u_q)_{\alpha,\beta} dx + \int_{\mathbb{R}^3} \mathbb{B}_{\gamma\delta,\alpha} \mathbb{A}_{\gamma\delta,\beta} (u_q)_{\alpha,\beta} dx \\
&= - \int_{\mathbb{R}^3} \mathbb{A}_{\gamma\delta,\alpha\beta} \mathbb{B}_{\gamma\delta,\beta} (u_q)_\alpha dx - \int_{\mathbb{R}^3} \mathbb{A}_{\gamma\delta,\alpha} \mathbb{B}_{\gamma\delta,\beta\beta} (u_q)_\alpha dx \\
&\quad - \int_{\mathbb{R}^3} \mathbb{B}_{\gamma\delta,\alpha\beta} \mathbb{A}_{\gamma\delta,\beta} (u_q)_\alpha dx - \int_{\mathbb{R}^3} \mathbb{B}_{\gamma\delta,\alpha} \mathbb{A}_{\gamma\delta,\beta\beta} (u_q)_\alpha dx \\
&= - \int_{\mathbb{R}^3} \mathbb{A}_{\gamma\delta,\alpha\beta} \mathbb{B}_{\gamma\delta,\beta} (u_q)_\alpha dx - \int_{\mathbb{R}^3} \mathbb{A}_{\gamma\delta,\alpha} \mathbb{B}_{\gamma\delta,\beta\beta} (u_q)_\alpha dx \\
&\quad + \int_{\mathbb{R}^3} \mathbb{B}_{\gamma\delta,\beta} \mathbb{A}_{\gamma\delta,\alpha\beta} (u_q)_\alpha dx + \int_{\mathbb{R}^3} \mathbb{B}_{\gamma\delta,\beta} \mathbb{A}_{\gamma\delta,\beta} (u_q)_{\alpha,\alpha} dx - \int_{\mathbb{R}^3} \mathbb{B}_{\gamma\delta,\alpha} \mathbb{A}_{\gamma\delta,\beta\beta} (u_q)_\alpha dx \\
&= - \int_{\mathbb{R}^3} (u_q \cdot \nabla) \mathbb{Q}_{\leq q-2} \cdot \Delta \mathbb{Q}_q dx - \int_{\mathbb{R}^3} (u_q \cdot \nabla) \mathbb{Q}_q \Delta \mathbb{Q}_{\leq q-2} dx.
\end{aligned}$$

Thus, identity (3.23) follows immediately.

We start now estimating the terms in $J_2 + K$. It follows from (3.23) that

$$\begin{aligned}
& |J_{212} + J_{222} + K_{22}| \\
&\leq \sum_{q>Q} \lambda_q^{2s} \int_{\mathbb{R}^3} |(u_q \cdot \nabla) \mathbb{Q}_q \Delta \mathbb{Q}_{\leq q-2}| dx + \sum_{q\leq Q} \lambda_q^{2s} \int_{\mathbb{R}^3} |(u_q \cdot \nabla) \mathbb{Q}_q \Delta \mathbb{Q}_{\leq q-2}| dx \\
&\lesssim \sum_{q>Q} \lambda_q^{2s} \|u_q\|_r \|\nabla \mathbb{Q}_q\|_2 \sum_{p\leq q-2} \|\Delta \mathbb{Q}_p\|_{\frac{2r}{r-2}} + \sum_{q\leq Q} \lambda_q^{2s} \|u_q\|_\infty \|\nabla \mathbb{Q}_q\|_2 \sum_{p\leq q-2} \|\Delta \mathbb{Q}_p\|_2 \\
&\lesssim c_r \mu \sum_{q>Q} \lambda_q^{2s+2-\frac{3}{r}} \|\mathbb{Q}_q\|_2 \sum_{p\leq q-2} \lambda_p^{2+\frac{3}{r}} \|\mathbb{Q}_p\|_2 + f(t) \sum_{q\leq Q} \lambda_q^{2s} \|\mathbb{Q}_q\|_2 \sum_{p\leq q-2} \lambda_p^2 \|\mathbb{Q}_p\|_2 \\
&\lesssim c_r \mu \sum_{q>Q} \lambda_q^{s+2} \|\mathbb{Q}_q\|_2 \sum_{p\leq q-2} \lambda_p^{s+2} \|\mathbb{Q}_p\|_2 \lambda_{q-p}^{s-\frac{3}{r}} + f(t) \sum_{q\leq Q} \lambda_q^{s+1} \|\mathbb{Q}_q\|_2 \sum_{p\leq q-2} \lambda_p^{s+1} \|\mathbb{Q}_p\|_2 \lambda_{q-p}^{s-1} \\
&\lesssim c_r \mu \sum_{q>Q} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2 + f(t) \sum_{q\leq Q} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2
\end{aligned}$$

for $s < \frac{3}{r}$ and $s < 1$.

Some of the rest terms are relatively easy. For instance, $J_{211} + J_{221}$ can be estimated similarly as J_{111} , while $J_{213} + J_{223}$ similarly as J_{113} , K_{13} and K_{23} similarly as L_{13} . Also, recalling that one of the benefits of commutator is to move derivatives onto low frequency parts (see Lemma 2.7), we observe K_{11} and K_{21} can be handled in an analogous way as L_2 and L_{11} , respectively.

The term J_{23} is estimated as follows,

$$\begin{aligned}
|J_{23}| &\leq \left| \sum_{q \geq -1} \lambda_q^{2s} \sum_{p \geq q-2} \int_{\mathbb{R}^3} \Delta_q (\nabla \mathbb{Q}_p \otimes \nabla \tilde{\mathbb{Q}}_p) \nabla u_q \, dx \right| \\
&\leq \sum_{q > Q} \lambda_q^{2s} \sum_{p \geq q-2} \int_{\mathbb{R}^3} \left| \Delta_q (\nabla \mathbb{Q}_p \otimes \nabla \tilde{\mathbb{Q}}_p) \nabla u_q \right| \, dx \\
&\quad + \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \sum_{p \geq q-2} \int_{\mathbb{R}^3} \left| \Delta_q (\nabla \mathbb{Q}_p \otimes \nabla \tilde{\mathbb{Q}}_p) \nabla u_q \right| \, dx \\
&\equiv J_{231} + J_{232},
\end{aligned}$$

with

$$\begin{aligned}
|J_{231}| &\lesssim \sum_{q > Q} \|\nabla u_q\|_\infty \sum_{p \geq q-2} \lambda_p^{2s} \|\nabla \mathbb{Q}_p\|_2^2 \\
&\lesssim c_r \mu \sum_{q > Q} \lambda_q^2 \sum_{p \geq q-2} \lambda_p^{2s+2} \|\mathbb{Q}_p\|_2^2 \\
&\lesssim c_r \mu \sum_{p > Q-2} \lambda_p^{2s+4} \|\mathbb{Q}_p\|_2^2 \sum_{q \leq p+2} \lambda_{q-p}^2 \\
&\lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
|J_{232}| &\lesssim \sum_{-1 \leq q \leq Q} \|\nabla u_q\|_\infty \sum_{p \geq q-2} \lambda_p^{2s} \|\nabla \mathbb{Q}_p\|_2^2 \\
&\lesssim f(t) \sum_{-1 \leq q \leq Q} \sum_{p \geq q-2} \lambda_p^{2s+2} \|\mathbb{Q}_p\|_2^2 \\
&\lesssim f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.
\end{aligned}$$

Noticing the cancelation

$$\int_{\mathbb{R}^3} (u \cdot \nabla) \mathbb{Q} \cdot \Delta \mathbb{Q} \, dx + \int_{\mathbb{R}^3} (\nabla \mathbb{Q} \otimes \nabla \mathbb{Q}) \nabla u \, dx = 0,$$

one can write $K_{12} = - \int_{\mathbb{R}^3} (\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_q) \nabla u_{\leq q-2} \, dx$. Therefore, we infer

$$\begin{aligned}
|K_{12}| &\leq \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} |(\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_q) \nabla u_{\leq q-2}| \, dx \\
&\leq \sum_{q \leq Q+2} \lambda_q^{2s} \int_{\mathbb{R}^3} |(\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_q) \nabla u_{\leq q-2}| \, dx \\
&\quad + \sum_{q > Q+2} \lambda_q^{2s} \int_{\mathbb{R}^3} |(\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_q) \nabla u_{\leq Q}| \, dx \\
&\quad + \sum_{q > Q+2} \lambda_q^{2s} \int_{\mathbb{R}^3} |(\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_q) \nabla u_{(Q, q-2]}| \, dx \\
&\equiv K_{121} + K_{122} + K_{123},
\end{aligned}$$

with

$$\begin{aligned}
|K_{121}| &\leq \sum_{q \leq Q+2} \lambda_q^{2s} \int_{\mathbb{R}^3} |(\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_q) \nabla u_{\leq q-2}| dx \\
&\lesssim \sum_{q \leq Q+2} \lambda_q^{2s} \|\nabla u_{\leq q-2}\|_\infty \|\nabla \mathbb{Q}\|_2^2 \\
&\lesssim Q f(t) \sum_{q \leq Q+2} \lambda_q^{2s+2} \|\mathbb{Q}\|_2^2;
\end{aligned}$$

$$\begin{aligned}
|K_{122}| &\leq \sum_{q > Q+2} \lambda_q^{2s} \int_{\mathbb{R}^3} |(\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_q) \nabla u_{\leq Q}| dx \\
&\lesssim \sum_{q > Q+2} \lambda_q^{2s} \|\nabla u_{\leq Q}\|_\infty \|\nabla \mathbb{Q}\|_2^2 \\
&\lesssim Q f(t) \sum_{q > Q+2} \lambda_q^{2s+2} \|\mathbb{Q}\|_2^2;
\end{aligned}$$

and

$$\begin{aligned}
|K_{123}| &\leq \sum_{q > Q+2} \lambda_q^{2s} \int_{\mathbb{R}^3} |(\nabla \mathbb{Q}_q \otimes \nabla \mathbb{Q}_q) \nabla u_{(Q, q-2]}| dx \\
&\lesssim \sum_{q > Q+2} \lambda_q^{2s} \|\nabla u_{(Q, q-2]}\|_\infty \|\nabla \mathbb{Q}\|_2^2 \\
&\lesssim \sum_{q > Q+2} \lambda_q^{2s+2} \|\mathbb{Q}\|_2^2 \sum_{Q < p \leq q-2} \lambda_p \|u_p\|_\infty \\
&\lesssim c_r \mu \sum_{q > Q+2} \lambda_q^{2s+2} \|\mathbb{Q}\|_2^2 \sum_{Q < p \leq q-2} \lambda_p^2 \\
&\lesssim c_r \mu \sum_{q > Q+2} \lambda_q^{2s+4} \|\mathbb{Q}\|_2^2.
\end{aligned}$$

To estimate K_3 , we use integration by parts first, and then split it as

$$\begin{aligned}
|K_3| &\leq \left| \sum_{p \geq -1} \lambda_q^{2s} \sum_{-1 \leq q \leq p+2} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla) \tilde{\mathbb{Q}}_p \Delta \mathbb{Q}_q dx \right| \\
&\leq \sum_{p > Q} \lambda_q^{2s} \sum_{-1 \leq q \leq p+2} \int_{\mathbb{R}^3} |\Delta_q (u_p \cdot \nabla) \tilde{\mathbb{Q}}_p \Delta \mathbb{Q}_q| dx \\
&\quad + \sum_{-1 \leq p \leq Q} \lambda_q^{2s} \sum_{-1 \leq q \leq p+2} \int_{\mathbb{R}^3} |\Delta_q (u_p \cdot \nabla) \tilde{\mathbb{Q}}_p \Delta \mathbb{Q}_q| dx \\
&\equiv K_{31} + K_{32},
\end{aligned}$$

with

$$\begin{aligned}
|K_{31}| &\lesssim \sum_{p>Q} \|u_p\|_\infty \|\nabla \mathbb{Q}_p\|_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s} \|\Delta \mathbb{Q}_q\|_2 \\
&\lesssim c_r \mu \sum_{p>Q} \lambda_p \|\nabla \mathbb{Q}_p\|_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2 \\
&\lesssim c_r \mu \sum_{p>Q} \lambda_p^{s+2} \|\mathbb{Q}_p\|_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{s+2} \|\mathbb{Q}_q\|_2 \lambda_{q-p}^s \\
&\lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2;
\end{aligned}$$

and

$$\begin{aligned}
|K_{32}| &\lesssim \sum_{-1 \leq p \leq Q} \|u_p\|_\infty \|\nabla \mathbb{Q}_p\|_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s} \|\Delta \mathbb{Q}_q\|_2 \\
&\lesssim f(t) \sum_{-1 \leq p \leq Q} \|\mathbb{Q}_p\|_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2 \\
&\lesssim f(t) \sum_{-1 \leq p \leq Q} \lambda_p^{s+1} \|\mathbb{Q}_p\|_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{s+1} \|\mathbb{Q}_q\|_2 \lambda_{q-p}^{s+1} \\
&\lesssim f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.
\end{aligned}$$

Following the above analysis and computation, we obtain

$$(3.24) \quad |J_2| + |K| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+4} \|\mathbb{Q}_q\|_2^2 + Q f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|\mathbb{Q}_q\|_2^2.$$

Combining (3.18)–(3.22) and (3.24) yields that for some small enough constant c_r with $2 \leq r < \frac{3}{s}$ and $\frac{1}{2} < s < 1$

$$\frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2s} (\|u_q\|_2^2 + \|\nabla \mathbb{Q}_q\|_2^2) \lesssim (Q(t)f(t) + 1) \sum_{q \geq -1} \lambda_q^{2s} (\|u_q\|_2^2 + \|\nabla \mathbb{Q}_q\|_2^2),$$

i.e., there exists an adimensional constant $C = C(r, \nu, \mu, s)$, such that

$$(3.25) \quad \frac{d}{dt} (\|u\|_{\dot{H}^s}^2 + \|\nabla \mathbb{Q}\|_{\dot{H}^s}^2) \leq C (Q(t)f(t) + 1) (\|u\|_{\dot{H}^s}^2 + \|\nabla \mathbb{Q}\|_{\dot{H}^s}^2).$$

We claim that, for $t > 0$

$$(3.26) \quad Q(t) \leq C(\nu, \mu, s) (1 + \log \|u(t)\|_{\dot{H}^s})$$

Indeed, it follows from (3.17) and Bernstein's inequality that

$$\Lambda(t) \leq (c_r \min\{\nu, \mu\})^{-1} \Lambda(t)^{\frac{2}{r}} \|u_{\mathbb{Q}(t)}(t)\|_r \leq (c_r \min\{\nu, \mu\})^{-1} \Lambda(t)^{\frac{2}{r}} \|u_{\mathbb{Q}(t)}(t)\|_2.$$

Thus, we obtain

$$\Lambda^{s-\frac{1}{2}}(t) \leq (c_r \min\{\nu, \mu\})^{-1} \|u(t)\|_{\dot{H}^s}.$$

Since $s > \frac{1}{2}$, (3.26) follows immediately. Combining (3.25) and (3.26) yields

$$\frac{d}{dt} (\|u\|_{\dot{H}^s}^2 + \|\nabla \mathbb{Q}\|_{\dot{H}^s}^2) \leq C(\nu, \mu, r, s) f(t) (1 + \log \|u(t)\|_{\dot{H}^s}) (\|u\|_{\dot{H}^s}^2 + \|\nabla \mathbb{Q}\|_{\dot{H}^s}^2).$$

Therefore, due to the assumption $f \in L^1(0, T)$, applying Grönwall's inequality to the above inequality gives us that $\|u(t)\|_{\dot{H}^s}^2 + \|\nabla \mathbb{Q}(t)\|_{\dot{H}^s}^2$ is bounded on $[0, T)$. It concludes the proof of Theorem 1.1.

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